

Logarithmic Integrals, Polylogarithmic Integrals and Euler sums

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Abstract

Relations among integrals of logarithms, polylogarithms and Euler sums are presented. A unifying element being the introduction of Nielsen's generalized polylogarithms.

Keywords: Logarithmic integrals, polylogarithmic integrals, Euler sums, Nielsen's generalized polylogarithms, multiple polylogarithms

1 Introduction

Recently a number of integrals and Euler sums of interest in Physics and Quantum Chemistry have been shown to be interrelated [1]. When calculating radiative corrections in Quantum Field Theory, multi-dimensional Feynman integrals which lead to these sums are encountered. For typical applications to Quantum Field Theory, see Cheng and Wu [2]. A common theme in these works has been the presence of Nielsen's generalized polylogarithm functions [3]. In this article we continue these investigations and provide a number of new identities involving integrals of logarithms, polylogarithms, Euler sums and Nielsen functions.

2 Polylogarithm Integrals

Integrals containing the product of two polylogarithmic functions $Li_r(t)$ [4] of orders p and q and having the forms ($p, q \geq 1$)

$$I_+(p, q) = \int_0^1 \frac{Li_p(t) Li_q(t) dt}{t} = I_+(q, p), \quad (1)$$

$$I_{-}(p, q) = \int_0^1 \frac{Li_p(-t) Li_q(-t) dt}{t} = I_{-}(q, p), \quad (2)$$

$$I_{\pm}(p, q) = \int_0^1 \frac{Li_p(t) Li_q(-t) dt}{t}, \quad (3)$$

arise in a number of physical applications [5] and are of interest in themselves. The integral $I_{+}(p, q)$ has been studied by Frietas [6]. We note the p, q symmetry displayed in Eqs. (1) and (2). Integration by parts for any of the integrals above allows one to obtain the following partial difference equation

$$I(p, q) + I(p+1, q-1) = R(p+1, q). \quad (4)$$

A more symmetric form of this difference equation being

$$I(p, q-1) + I(p-1, q) = R(p, q), \quad (5)$$

where $I(p, q)$ stands for any of the integrals in Eqs. (1-3), and $R(p, q)$ stands for one of the three corresponding quantities

$$R_{+}(p, q) = Li_p(1) Li_q(1) = \zeta(p) \zeta(q),$$

$$R_{-}(p, q) = Li_p(-1) Li_q(-1) = (2^{1-p} - 1) \zeta(p) (2^{1-q} - 1) \zeta(q),$$

$$R_{\pm}(p, q) = Li_p(1) Li_q(-1) = \zeta(p) (2^{1-q} - 1) \zeta(q),$$

and $\zeta(z)$ is the Riemann zeta function. We note that in the special case where z approaches 1 that

$$\lim_{z \rightarrow 1} (2^{1-z} - 1) \zeta(z) = -\ln(2).$$

A set of solutions for the difference equation Eq. (5) is given by

$$I(p+n, q-n) = (-1)^n \left[I(p, q) - \sum_{k=0}^{n-1} (-1)^k R(p+k+1, q-k) \right]. \quad (6)$$

Proof : If p is replaced by $p+k$ and q is replaced by $q-k$ in Eq. (4) we get

$$I(p+k, q-k) + I(p+k+1, q-k-1) = R(p+k+1, q-k).$$

Multiplying this by $(-1)^k$ and summing k over n terms produces

$$\sum_{k=0}^{n-1} (-1)^k I(p+k, q-k) + \sum_{k=0}^{n-1} (-1)^k I(p+k+1, q-k-1) = \sum_{k=0}^{n-1} (-1)^k R(p+k+1, q-k).$$

Writing out the *last* term in the second sum and the *first* term in the first sum on the lhs of this equation produces

$$(-1)^{n-1} I(p+n, q-n) + I(p, q) = \sum_{k=0}^{n-1} (-1)^k R(p+k+1, q-k),$$

the remaining sums having cancelled. Rearrangement of this equation gives the desired result.

3 Special cases of the $I(p, q)$ integrals

3.0.1 The $I_+(p, q)$ and $I_-(p, q)$ integrals

The odd case $q - p = 2n - 1$ In the case of $I_+(p, q)$, and $I_-(p, q)$ we note that when the integrals' parameters q and p are set equal and substituted in Eq. (5) that $I(p, p - 1) = \frac{1}{2}R(p, p)$, or more usefully

$$I(p + n, p + n - 1) = \frac{1}{2}R(p + n, p + n), \quad (7)$$

i.e. the I integrals can be written in closed form. This follows directly from the p, q symmetry of these integrals and Eq. (5). Consequently, the *family* of all such integrals where q and p differ by an *odd* integer can likewise be expressed in closed form. To see this we set $q = p + 2n - 1$ in Eq. (6) and solve for $I(p, p + 2n - 1)$ having used Eq. (7) and noting that the last term in the sum also contains the quantity $R(p + n, p + n)$. We find

$$I(p, p + 2n - 1) = \frac{1}{2}(-1)^{n+1}R(p + n, p + n) + \sum_{k=0}^{n-2} (-1)^k R(p + k + 1, p + 2n - 1 - k), \quad (8)$$

that is, these integrals are given by a finite sum of zeta functions. Using the p, q values associated with the series defined by $p + q = 5$ we have for either $I_+(p, q)$ or $I_-(p, q)$, the examples

$$I(1, 4) = -\frac{1}{2}R(3, 3) + R(2, 4),$$

$$I(2, 3) = \frac{1}{2}R(3, 3).$$

The even case $q - p = 2n$ In the case where q and p are set equal in Eq. (4), the integrals $I_+(p, q)$ and $I_-(p, q)$ have the property

$$I(p + 1, p - 1) = R(p + 1, p) - I(p, p).$$

All of the integrals in these families can be related to the ones which contain the square of a single polylogarithmic function. Setting $q = p + 2n$ in Eq. (6) we get

$$I(p, p + 2n) = (-1)^n I(p + n, p + n) + \sum_{k=0}^{n-1} (-1)^k R(p + k + 1, p + 2n - k). \quad (9)$$

By way of example, for the series associated with $p + q = 6$ we have for $I_+(p, q)$ or $I_-(p, q)$

$$I(1, 5) = I(3, 3) + R(2, 5) - R(3, 4),$$

$$I(2, 4) = -I(3, 3) + R(3, 4).$$

3.0.2 The $I_{\pm}(p, q)$ integral

The odd case No simple relationship such as Eq. (7) exists for $I_{\pm}(p, p-1)$, instead there is the slightly more complicated form obtained from Eq. (5) where p and q have been set equal and p has been replaced by $p+n$. We get

$$I_{\pm}(p+n, p+n-1) + I_{\pm}(p+n-1, p+n) = R_{\pm}(p+n, p+n).$$

Setting $q = p+2n-1$ and substituting $I_{\pm}(p+n, p+n-1)$ into Eq. (6) we get upon writing out the last term in the sum

$$I_{\pm}(p, p+2n-1) = (-1)^{n+1} I_{\pm}(p+n-1, p+n) + \sum_{k=0}^{n-2} (-1)^k R_{\pm}(p+k+1, p+2n-1-k).$$

Here we see that knowledge of the integrals $I_{\pm}(p+n-1, p+n)$ is required to compute $I_{\pm}(p, p+2n-1)$.

The even case Relations similar to Eq. (9) exist for the $I_{\pm}(p, q)$ integrals. If q and p are set equal in Eq. (4)

$$I_{\pm}(p+1, p-1) = R_{\pm}(p+1, p) - I_{\pm}(p, p).$$

The integrals $I_{\pm}(p+1, p-1)$ are seen to be related to $I_{\pm}(p, p)$ which contains two polylogarithmic functions of order p but with arguments differing in sign. Setting $q = p+2n$ in Eq. (6) we get a result which has exactly the same form as that for $I_{+}(p, q)$, or $I_{-}(p, q)$ as shown in Eq. (9) i.e.

$$I_{\pm}(p, p+2n) = (-1)^n I_{\pm}(p+n, p+n) + \sum_{k=0}^{n-1} (-1)^k R_{\pm}(p+k+1, p+2n-k).$$

4 Polylogarithm I integrals with low order

In cases with low order, some special integrals arise i.e. for $q = 0, 1$ we have

$$\begin{aligned} \int_0^1 \frac{Li_p(t)}{1+t} dt &= -I_{\pm}(p, 0), \\ \int_0^1 \frac{Li_p(-t)}{1+t} dt &= -I_{-}(p, 0), \\ \int_0^1 \frac{Li_p(t) - Li_p(1)}{1-t} dt &= -I_{+}(1, p-1), \\ \int_0^1 \frac{Li_p(-t) - Li_p(-1)}{1-t} dt &= -I_{\pm}(1, p-1). \end{aligned}$$

Integrals of the sort shown above arise in the calculation of hadronic heavy quark production [7]. These are examples of *multiple polylogarithms* $Li_{m_1}, \dots, m_n(x_1, \dots, x_n)$ as defined by

$$Li_{m_1, \dots, m_n}(x_1, \dots, x_n) = \sum_{k_n > \dots > k_2 > k_1 > 0} \frac{x_n^{k_n} \dots x_2^{k_2} x_1^{k_1}}{k_n^{m_n} \dots k_2^{m_2} k_1^{m_1}}.$$

Expressing the integrals above in these terms we have

$$\begin{aligned} \int_0^1 \frac{Li_p(\pm t)}{1+t} dt &= -Li_{p,1}(\mp 1, -1), \\ \int_0^1 \frac{Li_p(t) - Li_p(1)}{1-t} dt &= Li_{p,1}(1, 1) - \zeta(p+1), \\ \int_0^1 \frac{Li_p(-t) - Li_p(-1)}{1-t} dt &= Li_{p,1}(1, -1) + (1 - 2^{-p})\zeta(p+1). \end{aligned}$$

5 Infinite series representations for the I integrals

As noted above, for an exact calculation of the $I_+(p, q)$, $I_-(p, q)$ and $I_\pm(p, q)$ integrals, closed form expressions for the integrals $I(r, r)$ are needed for either even or odd values of r . In the case of the quantities $I_\pm(p, q)$, integrals of the form $I_\pm(r, r+1)$ are also required. As will be seen below, it is possible to express these integrals as infinite series whose sums can be found in closed form in at least some cases.

We will begin with $I_-(p, q)$. Expanding one of the polylogarithm functions say $Li_q(-t)$ in a power series within its integrand, we get

$$I_-(p, q) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^q} \int_0^1 Li_p(-t) t^{k-1} dt.$$

The moment integrals appearing in this sum can be obtained by repeated integration by parts and we have the general result (with empty sum for $p=1$)

$$\int_0^1 Li_p(-t) t^{k-1} dt = \frac{(-1)^p}{k^p} [\psi(k+1) - \psi(k/2+1)] + \sum_{\mu=2}^p \frac{(-1)^{p-\mu}}{k^{p+1-\mu}} (2^{1-\mu} - 1) \zeta(\mu),$$

where $\psi(z)$ is the Psi (Digamma) function. Boyadzhiev has computed the moments of $Li_p(t)$ [8]. The integral $I_-(p, q)$ then becomes

$$I_-(p, q) = (-1)^p \sum_{\mu=2}^p (-1)^\mu (2^{1-\mu} - 1) \zeta(\mu) (2^{\mu-p-q} - 1) \zeta(p+q+1-\mu)$$

$$+ (-1)^p \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{p+q}} [\psi(k+1) - \psi(k/2+1)].$$

The infinite sums in this expression can be represented in a more convenient form by writing out the $\psi(k/2+1)$ portion of the latter sum in terms of its even and odd values of k . We get

$$\begin{aligned} I_-(p, q) &= (-1)^p 2 \left[\ln(2) (2^{-p-q} - 1) \zeta(p+q) + (1 - 2^{-p-q-1}) \zeta(p+q+1) \right] \\ &\quad + (-1)^p \sum_{\mu=2}^p (-1)^\mu (2^{1-\mu} - 1) \zeta(\mu) (2^{\mu-p-q} - 1) \zeta(p+q+1-\mu) \\ &\quad + (-1)^p \left[\sum_{k=1}^{\infty} \frac{(-1)^k [\psi(k+1) + \gamma]}{k^{p+q}} - \sum_{k=1}^{\infty} \frac{[\psi(k+1) + \gamma]}{(2k)^{p+q}} + \sum_{k=0}^{\infty} \frac{[\psi(k+1/2) - \psi(1/2)]}{(2k+1)^{p+q}} \right], \end{aligned}$$

where γ is Euler's constant. In a similar way the integrals $I_+(p, q)$ and $I_\pm(p, q)$ are given by

$$\begin{aligned} I_+(p, q) &= (-1)^p \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) \zeta(p+q+1-\mu) + (-1)^{p+1} \sum_{k=1}^{\infty} \frac{1}{k^{p+q}} [\psi(k+1) + \gamma], \\ I_\pm(p, q) &= (-1)^p \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) (2^{\mu-p-q} - 1) \zeta(p+q+1-\mu) + (-1)^{p+1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{p+q}} [\psi(k+1) + \gamma]. \end{aligned}$$

Although not manifestly obvious, the p, q symmetry is preserved in the infinite sum relations for $I_+(p, q)$ and $I_-(p, q)$.

6 Closed forms for the infinite series

In the expressions for the I integrals given above, four infinite series containing Psi functions occur. We will show below that it is possible to give closed form expressions for two of them whereas in the remaining two, closed forms can be given only for *even* values of $p+q$.

We begin with a discussion of the Euler sum $S_+(r)$ of order r i.e.

$$S_+(r) = \sum_{k=1}^{\infty} \frac{1}{k^r} [\psi(k+1) + \gamma].$$

A closed form expression for this sum where $r \geq 2$ had originally been given by Euler i.e.

$$S_+(r) = \frac{1}{2}(r+2) \zeta(r+1) - \frac{1}{2} \sum_{\mu=1}^{r-2} \zeta(\mu+1) \zeta(r-\mu).$$

In the case of the *alternating* Euler sum $S_-(r)$ i.e.

$$S_-(r) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^r} [\psi(k+1) + \gamma], \quad (10)$$

the situation is more complicated. Rewriting the series in Eq. (10) as sums corresponding to the even and odd values of k respectively, we find

$$S_-(r) = \sum_{k=1}^{\infty} \frac{[\psi(2k+1) + \gamma]}{(2k)^r} - \sum_{k=0}^{\infty} \frac{[\psi(2k+2) + \gamma]}{(2k+1)^r}.$$

Each of the Psi functions in these sums can be replaced by functions of lower argument using the Psi function's duplication formulas. We get

$$\begin{aligned} S_-(r) &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+1/2)}{(2k)^r} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\psi(k+1)}{(2k)^r} + (\gamma + \ln(2)) 2^{-r} \zeta(r) \\ &\quad - \frac{1}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1)}{(2k+1)^r} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1/2)}{(2k+1)^r} - (\gamma + \ln(2)) (1-2^{-r}) \zeta(r) - (1-2^{-r-1}) \zeta(r+1). \end{aligned}$$

This expression can be simplified with the help of four known sums which we call the Jordan sums $\mathcal{J}_1(r)$, $\mathcal{J}_2(r)$; the Milgram sum $\mathcal{M}(r)$, and the sum $\mathcal{C}(r)$ where

$$\mathcal{J}_1(r) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{[\psi(k+1/2) - \psi(1/2)]}{(2k+1)^r}, \quad (11a)$$

$$\mathcal{J}_2(r) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{[\psi(k+1/2) - \psi(1/2)]}{(2k)^r}, \quad (11b)$$

$$\mathcal{M}(r) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \gamma]}{(2k+1)^r}, \quad (11c)$$

$$\mathcal{C}(r) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{[\psi(k+1) + \gamma]}{(2k)^r} = \frac{1}{2^{r+1}} S_+(r). \quad (11d)$$

The sum $S_-(r)$ can then be rewritten as

$$S_-(r) = \mathcal{J}_2(r) - \mathcal{J}_1(r) + \mathcal{C}(r) - \mathcal{M}(r) - (1-2^{-r-1}) \zeta(r+1).$$

Closed form expressions for the sums $\mathcal{J}_1(r)$ and $\mathcal{J}_2(r)$ have been given by Jordan [10] only for even values of r i.e.

$$\begin{aligned} \mathcal{J}_1(2n) &= -\frac{1}{2} (1-2^{-2n-1}) \zeta(2n+1) + \ln(2) (1-2^{-2n}) \zeta(2n) \\ &\quad - 2^{-2n-1} \sum_{\mu=1}^{n-1} (2^{2\mu} - 1) \zeta(2\mu) \zeta(2n+1-2\mu), \quad (n \geq 1), \end{aligned}$$

and

$$\mathcal{J}_2(2n) = \frac{1}{2}(1 - 2^{-2n-1}) \zeta(2n+1) - \sum_{\mu=1}^{n-1} (2^{-2\mu} - 2^{-2n-1}) \zeta(2\mu) \zeta(2n+1-2\mu).$$

The sum $\mathcal{M}(r)$ has been given by Milgram [9] for $r \geq 2$ in closed form (with empty sum for $r \leq 2$) as

$$\begin{aligned} \mathcal{M}(r) &= \frac{1}{2}r(1 - 2^{-r-1}) \zeta(r+1) - \ln(2)(1 - 2^{-r}) \zeta(r) \\ &\quad - \frac{1}{2(r-1)} \sum_{\mu=0}^{r-3} (\mu+1)(2^{\mu+2} - 1) \zeta(\mu+2)(2^{-\mu-1} - 2^{-r}) \zeta(r-1-\mu). \end{aligned}$$

This Milgram sum can be simplified considerably because of the reoccurrence of the zeta products within the sum. We get

$$\begin{aligned} \mathcal{M}(r) &= \frac{1}{2}r(1 - 2^{-r-1}) \zeta(r+1) - \ln(2)(1 - 2^{-r}) \zeta(r) \\ &\quad - \frac{1}{2} \sum_{\mu=0}^{(r-4)/2} (2^{\mu+2} - 1) \zeta(\mu+2)(2^{-\mu-1} - 2^{-r}) \zeta(r-1-\mu), \quad \text{for even } r \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(r) &= \frac{1}{2}r(1 - 2^{-r-1}) \zeta(r+1) - \ln(2)(1 - 2^{-r}) \zeta(r) - \frac{1}{2} \left[(1 - 2^{-\left(\frac{r+1}{2}\right)}) \zeta\left(\frac{r+1}{2}\right) \right]^2 \\ &\quad - \frac{1}{2} \sum_{\mu=0}^{(r-5)/2} (2^{\mu+2} - 1) \zeta(\mu+2)(2^{-\mu-1} - 2^{-r}) \zeta(r-1-\mu), \quad \text{for odd } r \end{aligned}$$

The closed form expression for the sum $\mathcal{C}(r)$, has already been given above as a multiple of $S_+(r)$.

Finally, the values of the I integrals can be written in terms of these sums as

$$\begin{aligned} I_+(p, q) &= (-1)^p \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) \zeta(p+q+1-\mu) + (-1)^{p+1} S_+(p+q), \\ I_\pm(p, q) &= (-1)^p \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) (2^{\mu-p-q} - 1) \zeta(p+q+1-\mu) + (-1)^{p+1} S_\pm(p+q), \\ I_-(p, q) &= (-1)^p 2 \left[\ln(2) (2^{-p-q} - 1) \zeta(p+q) + (1 - 2^{-p-q-1}) \zeta(p+q+1) \right] \\ &\quad + (-1)^p \sum_{\mu=2}^p (-1)^\mu (2^{1-\mu} - 1) \zeta(\mu) (2^{\mu-p-q} - 1) \zeta(p+q+1-\mu) \\ &\quad + (-1)^p [S_-(p+q) - 2\mathcal{C}(p+q) + 2\mathcal{J}_1(p+q)]. \end{aligned}$$

To summarize the situation, we have found that closed form expressions for $I_+(p, q)$ and $I_-(p, q)$ when $q = p + 2n - 1$ are given by Eq. (8). In the cases where $q = p + 2n$ the sums $S_+(2p + 2n)$ and $S_-(2p + 2n)$, which are known in closed form, are required for the calculation of $I_+(p, q)$ and $I_-(p, q)$.

In the case of $I_\pm(p, q)$ where $q = p + 2n - 1$ the sum $S_-(2p + 2n - 1)$ is required but this is not generally known in closed form. When $q = p + 2n$ where $S_-(2p + 2n)$ is required, the integral $I_\pm(p, 2p + 2n)$ is calculable in closed form.

6.1 Connection with Nielsen's polylogarithm functions

The alternating Euler sums $S_-(r)$ and the Jordan sums $\mathcal{J}_1(r)$ and $\mathcal{J}_2(r)$ can be expressed in terms of Nielsen's generalized polylogarithm functions [11] $\mathcal{S}_{n,p}(z)$ i.e.

$$\mathcal{S}_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\ln^{n-1}(x) \ln^p(1-zx)}{x} dx,$$

and the three special cases $\mathcal{S}_{n,1}(z) = Li_{n+1}(z)$, $\mathcal{S}_{n,p}(1) = s_{n,p}$ and $\mathcal{S}_{n,p}(-1) = \tilde{\sigma}_{n,p}$ where the latter two integrals are given by

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\ln^{n-1}(x) \ln^p(1-x)}{x} dx = s_{p,n},$$

and

$$\tilde{\sigma}_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\ln^{n-1}(x) \ln^p(1+x)}{x} dx.$$

In the case of $S_-(r)$ Coffey [13] has given the following integral representation for $S_-(r)$

$$S_-(r) = \frac{(-1)^r}{\Gamma(r)} \int_0^1 \left(\frac{1}{x} - \frac{1}{1+x} \right) \ln^{r-1}(x) \ln(1+x) dx,$$

which can be rewritten in terms of the polylogarithm function and Nielsen's generalized polylogarithm function $\mathcal{S}_{n,p}(z)$ i.e.

$$S_-(r) = Li_{r+1}(-1) + \mathcal{S}_{r-1,2}(-1) = (2^{-r} - 1)\zeta(r+1) + \tilde{\sigma}_{r-1,2}.$$

Both Lewin and Kolbig [11] have commented on their inability to express $\mathcal{S}_{n,p}(-1)$ in closed form for even values of n thus leaving the search for a closed form expressions for $S_-(2n+1) = (2^{-2n-1})\zeta(2n+2) + \tilde{\sigma}_{2n,2}$ as an open question.

It is interesting that closed form expressions for the sums $\mathcal{J}_1(r)$, and $\mathcal{J}_2(r)$ are not generally known for odd values of r except in the case of the two series

$$\frac{1}{2} \sum_{k=0}^{\infty} \frac{[\psi(k+1/2) - \psi(1/2)]}{(2k+1)^3} = \frac{23}{5760}\pi^4 + \frac{\pi^2}{24}\ln^2(2) - \frac{1}{24}\ln^4(2) - Li_4(1/2),$$

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{[\psi(k+1/2) - \psi(1/2)]}{(2k)^3} = \frac{7}{8} \ln(2) \zeta(3) - \frac{53}{5760} \pi^4 - \frac{\pi^2}{24} \ln^2(2) + \frac{1}{24} \ln^4(2) + Li_4(1/2), \quad (12)$$

which have been obtained (cf. Appendix A1) from work by Coffey [12]. The methods used to obtain these closed form expressions do not seem to apply for any other odd powers of r for the sums in question. Using the sums above, $S_-(3)$ is then given in closed form by

$$S_-(3) = \frac{7}{4} \ln(2) \zeta(3) - \frac{11}{360} \pi^4 - \frac{\pi^2}{12} \ln^2(2) + \frac{1}{12} \ln^4(2) + 2 Li_4(1/2).$$

It is also possible to find integral representation (cf. Appendix A1) for the sums $\mathcal{J}_1(r)$, $\mathcal{J}_2(r)$ and $\mathcal{C}(r)$ which are in turn expressible in terms of the quantities $s_{n,m}$ and $\tilde{\sigma}_{n,m}$. Finally the I integrals written in terms of the Nielsen integrals are given by

$$(-1)^p I_+(p, q) = \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) \zeta(p+q+1-\mu) - \zeta(p+q+1) - s_{p+q-1,2},$$

$$(-1)^p I_{\pm}(p, q) = \sum_{\mu=2}^p (-1)^\mu \zeta(\mu) (2^{\mu-p-q}-1) \zeta(p+q+1-\mu) - (2^{-p-q}-1) \zeta(p+q+1) - \tilde{\sigma}_{p+q-1,2},$$

$$\begin{aligned} (-1)^p I_-(p, q) &= 2 \ln(2) (2^{-p-q}-1) \zeta(p+q) + 2(1-2^{-p-q-1}) \zeta(p+q+1) \\ &\quad + \sum_{\mu=2}^p (-1)^\mu (2^{1-\mu}-1) \zeta(\mu) (2^{\mu-p-q}-1) \zeta(p+q+1-\mu) \\ &\quad + (1-2^{-p-q}) s_{p+q-1,2} - \zeta(p+q+1) - 2 \mathcal{M}(p+q). \end{aligned}$$

Written in these terms, the integrals I_+ and I_- are seen to be expressible in closed form in all cases whereas closed form expressions for I_{\pm} are obtainable for even $p+q$. Only in the case where $p+q=3$ are closed form expressions for I_{\pm} obtainable by the methods used here.

7 Logarithmic Integrals

The $i(n, m)$ integrals The integrals

$$i(n, m) = \int_0^1 \ln^n(x) \ln^m(1-x) dx = i(m, n),$$

are related to Nielsen's generalized polylogarithms $s_{n,p}$. Integration by parts yields the relations

$$\frac{i(n,m)}{n!m!} = (-1)^{m+n} - \sum_{\mu=0}^{m-1} (-1)^\mu \frac{i(n-1, m-\mu)}{(n-1)!(m-\mu)!} - (-1)^{m+n} \sum_{\mu=0}^{m-1} s_{n, m-\mu}.$$

If the quantities $i^*(n, m) = (-1)^{m+n} i(n, m)/m!n!$ are introduced into the equation above we get

$$i^*(n, m) = 1 + \sum_{\mu=0}^{m-1} i^*(n-1, m-\mu) - \sum_{\mu=0}^{m-1} s_{n, m-\mu}. \quad (13)$$

Writing out the lhs of this equation for the case $i^*(n, m-1)$ and subtracting it from Eq. (13) we get Pascal's triangular, three term partial difference equation

$$i^*(n, m) - i^*(n, m-1) - i^*(n-1, m) = -s_{n, m}. \quad (14)$$

This can be solved for $i^*(n, m)$ in closed form using the Laplace method of generating functions [14] with the initial condition $i^*(0, m) = 1$. The final result is

$$\begin{aligned} (-1)^{m+n} i(n, m) &= (m+n)! - m \cdot n \cdot (m+n-2)! \zeta(2) - m \cdot n! \sum_{\nu=2}^n \frac{(n-\nu+m-1)!}{(n-\nu)!} \zeta(\nu+1) \\ &\quad - n \cdot m! \sum_{\mu=2}^m \frac{(n-\mu+m-1)!}{(n-\mu)!} \zeta(\mu+1) - m! n! \sum_{\mu=2}^m \sum_{\nu=2}^n \frac{(n-\nu+m-\mu)!}{(n-\nu)!(m-\mu)!} s_{\nu, \mu}. \end{aligned}$$

The integrals $s_{n,p}$ have been given by Kolbig [15] in closed form. Alternatively, they can be quickly generated with a computer algebra system such as Maple or Mathematica using the Kolbig relation

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!} \left[\frac{\partial^{n+p-1}}{\partial \beta^{n-1} \partial \alpha^p} \left\{ \frac{1}{\beta} \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} \right\} \right]_{\alpha=\beta=0}.$$

Using this expression, a number of the $i(n, m)$ integrals have been obtained and are displayed in the table below.

Table 1. $i(n, m), 1 \leq n, m \leq 3$

$i(n, m)$	$(-1)^{m+n}(m+n)!$	π^2	π^4	π^6	$\zeta(3)$	$\zeta(5)$	$\pi^2\zeta(3)$	$\zeta^2(3)$
$i_{1,1}$	2	-1/6						
$i_{1,2}$	-6	1/3			2			
$i_{1,3}$	24	-1	-1/15		-6			
$i_{2,2}$	24	-4/3	-1/90		-12			
$i_{2,3}$	-120	6	1/6			24	-2	
$i_{3,3}$	720	-36	-1	-23/420	-216	-144	12	36

An alternate and more direct way to compute the $i(n, m)$ integrals is due to a suggestion by Milgram [16] i.e. noting the relation

$$\beta(\nu + 1, \mu + 1) = \int_0^1 x^\mu (1 - x)^\nu dx,$$

differentiation n times with respect to μ and m times with respect to ν and evaluated at $\mu = \nu = 0$ give

$$\left[\frac{\partial^{m+n}}{\partial \nu^m \partial \mu^n} \beta(\nu + 1, \mu + 1) \right]_{\mu=\nu=0} = i(m, n)$$

As a final remark, we note that the three term partial difference equation Eq. (14) can alternately be viewed as a way to compute the $s_{n, m}$ integrals given independently computed values of the $i(n, m)$ integrals.

7.0.1 The $h(n, m)$ integrals

In a similar way the integrals defined by

$$h(n, m) = \int_0^1 \ln^n(x) \ln^m(1 + x) dx,$$

can be integrated by parts to give

$$\frac{h(n, m)}{n! m!} = (-1)^{m+n} + \sum_{\mu=0}^{m-1} (-1)^{n-1+m-\mu} \frac{h(n-1, m-\mu)}{(n-1)!(m-\mu)!} + (-1)^{m+n} \sum_{\mu=0}^{m-1} \tilde{\sigma}_{n, m-\mu},$$

which upon introducing the quantities $h^*(n, m) = (-1)^{m+n} h(n, m)/n! m!$ gives

$$h^*(n, m) = 1 + \sum_{\mu=0}^{m-1} h^*(n-1, m-\mu) + \sum_{\mu=0}^{m-1} \tilde{\sigma}_{n, m-\mu}.$$

As shown above, a three term partial difference equation for the quantities $h^*(n, m)$ can similarly be obtained i.e.

$$h^*(n, m) - h^*(n, m-1) - h^*(n-1, m) = \tilde{\sigma}_{n, m}.$$

In this case, the method of generating functions together with the initial conditions $h(0, m) = -1 + 2 e_m(-\ln(2))$, where $e_m(x)$ is the truncated exponential function, gives the solution

$$\begin{aligned} (-1)^{n+m} h(n, m) &= -(n+m)! + 2n! [-\ln(2)]^m {}_2F_2([-m, n+1], [], 1/\ln(2)) \\ &+ m! n! \sum_{\mu=1}^m \sum_{\nu=1}^n \frac{(n-\nu+m-\mu)!}{(n-\nu)!(m-\mu)!} \tilde{\sigma}_{\nu, \mu}, \end{aligned}$$

and where ${}_2F_2$ is a Gauss generalized hypergeometric function which reduces to a polynomial for the values of the parameters n, m encountered here. We have

$${}_2F_2([-m, n+1], [], 1/\ln(2)) = \frac{m!}{n!} \sum_{k=0}^m \frac{(n+k)!}{(m-k)!} \frac{[-1/\ln(2)]^k}{k!},$$

so that

$$\begin{aligned} (-1)^{n+m} h(n, m) &= (n+m)! + 2m! \sum_{\mu=1}^m \frac{(m+n-\mu)!}{(m-\mu)!} \frac{[-\ln(2)]^\mu}{\mu!} \\ &\quad + m! n! \sum_{\mu=1}^m \sum_{\nu=1}^n \frac{(n-\nu+m-\mu)!}{(n-\nu)!(m-\mu)!} \tilde{\sigma}_{\nu, \mu}. \end{aligned}$$

Values for the first few $h(n, m)$ integrals are given below.

$$h(1, 1) = 2 - 2\ln(2) - \frac{\pi^2}{12},$$

$$\underline{h(n, m), \quad n+m=3}$$

$$h(1, 2) = 6 - \frac{\pi^2}{6} + \frac{\zeta(3)}{4} - 8\ln(2) + 2\ln^2(2),$$

$$h(2, 1) = -6 + \frac{\pi^2}{6} + \frac{3}{2}\zeta(3) + 4\ln(2),$$

$$\underline{h(n, m), \quad n+m=4}$$

$$\begin{aligned} h(1, 3) &= 24 - 36\ln(2) + 12\ln^2(2) - 2\ln^3(2) + \ln^4(2)/4 - \pi^2/2 - \pi^4/15 \\ &\quad + 3\zeta(3)/4 - \pi^2\ln^2(2)/4 + 21\ln(2)\zeta(3)/4 + 6Li_4(1/2), \end{aligned}$$

$$\begin{aligned} h(2, 2) &= 24 - 24\ln(2) + 4\ln^2(2) + \ln^4(2)/3 - 2\pi^2/3 - \pi^4/12 \\ &\quad - 5\zeta(3)/2 - \pi^2\ln^2(2)/3 + 7\ln(2)\zeta(3) + 8Li_4(1/2), \end{aligned}$$

$$h(3, 1) = 24 - 12\ln(2) - 7\ln^4(2)/120 - \pi^2/2 - 9\zeta(3)/2,$$

$$\underline{h(n, m), \quad n+m=5}$$

$$\begin{aligned} h(1, 4) &= -120 + 192\ln(2) - 72\ln^2(2) + 16\ln^3(2) - 3\ln^4(2) + 2\pi^2 + 4\pi^4/15 \\ &\quad - 3\zeta(3) + \pi^2\ln^2(2) - 21\ln(2)\zeta(3) - 24Li_4(1/2), \end{aligned}$$

$$\begin{aligned} h(2, 3) &= -120 + 144\ln(2) - 36\ln^2(2) + 4\ln^3(2) - 3\ln^4(2)/2 + 4\ln^5(2)/5 + 3\pi^2 \\ &\quad + 23\pi^4/60 + [6 - \pi^2 - 63\ln(2)/2 + 21\ln^2(2)/2]\zeta(3) - 99\zeta(5)/8 \\ &\quad + (3/2 - 2\ln(2)/3)\pi^2\ln^2(2) + [24\ln(2) - 36]Li_4(1/2) + 24Li_5(1/2), \end{aligned}$$

$$\begin{aligned} h(3, 2) &= -120 + 96\ln(2) - 12\ln^2(2) - \ln^4(2) + 3\pi^2 + 11\pi^4/30 \\ &\quad + [33/2 - \pi^2 - 21\ln(2)]\zeta(3) + 87\zeta(5)/8 + \pi^2\ln^2(2) - 24Li_4(1/2), \end{aligned}$$

$$h(4, 1) = -120 + 48\ln(2) + 2\pi^2 + 7\pi^4/30 + 18\zeta(3) + 45\zeta(5)/2,$$

7.1 Relations between the $s_{n,p}$ and $\tilde{\sigma}_{n,p}$ integrals

The integrals $s_{n,p}$ and $\tilde{\sigma}_{n,p}$ are linearly related. These relations are worth examining since they shed some light on the problem of finding closed form expressions for the even values of n in the $\tilde{\sigma}_{n,p}$ integrals. To show this we note that

$$\left[\frac{\partial^{n+m}}{\partial \mu^n \partial \nu^m} \left(\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \right) \right]_{\mu=0, \nu=1} = \int_0^1 \frac{\ln^n(x/(1+x)) \ln^m(1/(1+x))}{x(1+x)} dx + \int_0^1 \frac{\ln^m(x/(1+x)) \ln^n(1/(1+x))}{(1+x)} dx,$$

where the integral representation for the Beta function

$$\beta(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} = \int_0^1 \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx,$$

has been used. Expanding the logarithm terms in the integrals above, together with the relations

$$\int_0^1 \frac{\ln^q(x) \ln^p(1+x)}{(1+x)} dx = - \left(\frac{q}{p+1} \right) \int_0^1 \frac{\ln^{q-1}(x) \ln^{p+1}(1+x)}{x} dx,$$

we find

$$(-1)^{n+m} s_{n+1, m} = \sum_{k=0}^n (-1)^k \binom{n+m-1-k}{m-1} \tilde{\sigma}_{k+1, n+m-k} + \sum_{k=0}^{m-1} (-1)^k \binom{n+m-1-k}{n} \tilde{\sigma}_{k+1, n+m-k},$$

or in its more symmetric form

$$(-1)^{n+m} s_{n, m} = \sum_{k=1}^n (-1)^k \binom{n+m-1-k}{m-1} \tilde{\sigma}_{k, n+m-k} + \sum_{k=1}^m (-1)^k \binom{n+m-1-k}{n-1} \tilde{\sigma}_{k, n+m-k}. \quad (15)$$

As previously pointed out by Kolbig [17], we see that for a given value of $m+n$ there are (for $m+n \geq 6$) more of the $\tilde{\sigma}_{\nu, \mu}$ integrals than there are equations, given the structure of the relations given above in Eq. (15). Closed form values for some of the $\tilde{\sigma}_{\nu, \mu}$ integrals are given in the tables below

$$\begin{aligned} \tilde{\sigma}_{1,1} &= -\pi^2/12 \\ \tilde{\sigma}_{1,2} &= \zeta(3)/8 \\ \tilde{\sigma}_{2,1} &= -3\zeta(3)/4 \end{aligned}$$

Table 2. $\tilde{\sigma}_{\nu, \mu}$, $\nu + \mu = 4$

$\tilde{\sigma}_{\nu, \mu}$	π^4	$\pi^2 \ln^2(2)$	$\ln^4(2)$	$\ln(2)\zeta(3)$	$Li_4(1/2)$
$\tilde{\sigma}_{1,3}$	$-1/90$	$-1/24$	$1/24$	$7/8$	1
$\tilde{\sigma}_{2,2}$	$-1/48$	$-1/12$	$1/12$	$7/4$	2
$\tilde{\sigma}_{3,1}$	$-7/720$				

Table 3. $\tilde{\sigma}_{\nu,\mu}$, $\nu + \mu = 5$

$\tilde{\sigma}_{\nu,\mu}$	$\pi^2\zeta(3)$	$\zeta(5)$	$\ln^2(2)\zeta(3)$	$\pi^2\ln^3(2)$	$\ln^5(2)$	$\ln(2)Li_4(1/2)$	$Li_5(1/2)$
$\tilde{\sigma}_{1,4}$		1	$-7/16$	$1/36$	$-1/30$	-1	-1
$\tilde{\sigma}_{2,3}$	$1/12$	$33/32$	$-7/8$	$1/18$	$-1/15$	-2	-2
$\tilde{\sigma}_{3,2}$	$1/12$	$-29/32$					
$\tilde{\sigma}_{4,1}$		$-15/16$					

In the case where $\nu + \mu = 6$, it is not possible to obtain closed form expressions for the corresponding $\tilde{\sigma}_{\nu,\mu}$ values using these methods. However the following relations can be found.

$$\begin{aligned}\tilde{\sigma}_{1,5} = & -\frac{1}{945}\pi^6 - \frac{1}{96}\pi^2\ln^4(2) + \frac{1}{72}\ln^6(2) + \frac{7}{28}\ln^3(2)\zeta(3) \\ & + \frac{1}{2}\ln^2(2)Li_4(1/2) + \ln(2)Li_5(1/2) + Li_6(1/2),\end{aligned}$$

$$\begin{aligned}2\tilde{\sigma}_{2,4} - \tilde{\sigma}_{4,2} = & -\frac{53}{15120}\pi^6 - \frac{1}{24}\pi^2\ln^4(2) + \frac{1}{18}\ln^6(2) + \frac{7}{12}\ln^3(2)\zeta(3) \\ & - \frac{1}{2}\zeta^2(3) + 2\ln^2(2)Li_4(1/2) + 4\ln(2)Li_5(1/2) + 4Li_6(1/2),\end{aligned}$$

$$2\tilde{\sigma}_{3,3} - 3\tilde{\sigma}_{4,2} = \frac{1}{1512}\pi^6 - \frac{1}{2}\zeta^2(3),$$

$$\tilde{\sigma}_{5,1} = -\frac{31}{30240}\pi^6.$$

A Appendix

A.1 Jordan sums with odd order r

It is possible to express the Jordan sum $\mathcal{J}_1(2n+1)$ with odd argument as the integral

$$\mathcal{J}_1(2n+1) = \frac{1}{4(2n)!} \int_0^1 \ln^{2n}(x) \ln\left(\frac{1+x}{1-x}\right) \left[\frac{1}{1-x} - \frac{1}{1+x}\right] dx.$$

This can be done with the help of an integral representation [18] for the Psi function in the numerator and the usual integral representation for the denominator within the summand of Eq. (11a) i.e.

$$\frac{\psi(k+1/2) - \psi(1/2)}{2(2k+1)^{2n+1}} = \frac{1}{2(2n)!} \int_0^\infty u^{2n} \exp(-[2k+1]u) du \int_0^1 \frac{(1-t^k)}{\sqrt{t}(1-t)} dt.$$

Summation over k followed by integration over t produces the desired integral representation for $\mathcal{J}_1(2n+1)$.

In the case where $n = 1$, $\mathcal{J}_1(3)$ contains the integrals

$$\begin{aligned} \int_0^1 \frac{\ln^2(x) \ln(1-x)}{1-x} dx &= -\frac{\pi^4}{180}, \\ \int_0^1 \frac{\ln^2(x) \ln(1+x)}{1-x} dx &= \frac{7}{2} \ln(2) \zeta(3) - \frac{19}{720} \pi^4, \\ \int_0^1 \frac{\ln^2(x) \ln(1-x)}{1+x} dx &= \frac{\pi^4}{90} + \frac{\pi^2}{6} \ln^2(2) - \frac{1}{6} \ln^4(2) - 4Li_4(1/2), \\ \int_0^1 \frac{\ln^2(x) \ln(1+x)}{1+x} dx &= 4Li_4(1/2) - \frac{\pi^2}{24} - \frac{\pi^2}{6} \ln^2(2) + \frac{1}{6} \ln^4(2) + \frac{7}{2} \ln(2) \zeta(3), \end{aligned} \tag{16}$$

which have been given by Gastmann [21], Lewin [19] and Coffey [13]. These have been used to produce the sums given above in Eq. (12).

Using the same methods employed above in Eq. (11b), the integral representation for $\mathcal{J}_2(2n+1)$ is given by

$$\mathcal{J}_2(2n+1) = \frac{1}{4(2n)!} \int_0^1 \ln^{2n}(x) \ln\left(\frac{1+x}{1-x}\right) \left[\frac{1}{1-x} + \frac{1}{1+x} \right] dx,$$

where we note the difference in signs within the integrands of the \mathcal{J}_1 and \mathcal{J}_2 integral representations. This integral upon expansion can be rewritten as

$$\mathcal{J}_2(2n+1) = \frac{1}{4(2n)!} \int_0^1 \ln^{2n}(x) \left[\frac{\ln(1+x)}{(1+x)} - \frac{\ln(1-x)}{(1-x)} - \frac{d}{dx} \ln(1+x) \ln(1-x) \right] dx.$$

The first two integrals are related to Nielsen's generalized polylogarithms $s_{n,p}$ and $\tilde{\sigma}_{n,p}$ and we get

$$\mathcal{J}_2(2n+1) = \frac{1}{4} [s_{2n,2} + \sigma_{2n,2}] - \frac{1}{4(2n)!} \int_0^1 \ln^{2n}(x) \left[\frac{d}{dx} \ln(1+x) \ln(1-x) \right] dx.$$

The remaining integration within the \mathcal{J}_2 integral representation can be carried out using the expansion

$$-\frac{d}{dx} \ln(1+x) \ln(1-x) = \sum_{k=1}^{\infty} [2\ln(2) - \psi(k) + \psi(k+1/2)] x^{2k-1}.$$

We get

$$\frac{1}{4(2n)!} \int_0^1 \ln^{2n}(x) \left[-\frac{d}{dx} \ln(1+x) \ln(1-x) \right] dx = \frac{1}{4} \sum_{k=1}^{\infty} \frac{[2\ln(2) - \psi(k) + \psi(k+1/2)]}{(2k)^{2n+1}}.$$

The rhs of this equation can be written in terms of the sums defined above as

$$\frac{1}{2} [\mathcal{J}_2(2n+1) - \mathcal{C}(2n+1) + 2^{-2n-2}\zeta(2n+2)].$$

The combined integrals give $\mathcal{J}_2(2n+1)$ as

$$\mathcal{J}_2(2n+1) = \frac{1}{2}[s_{2n,2} + \tilde{\sigma}_{2n,2}] - \mathcal{C}(2n+1) + 2^{-2n-2}\zeta(2n+2).$$

In a similar way, the sum $\mathcal{J}_1(2n+1)$ can be expressed in terms of Nielsen's polylogarithms as follows. Adding the integral representations for \mathcal{J}_1 and \mathcal{J}_2 we get

$$\mathcal{J}_1(2n+1) + \mathcal{J}_2(2n+1) = \frac{1}{2(2n)!} \int_0^1 \ln^{2n}(x) \ln\left(\frac{1+x}{1-x}\right) \frac{dx}{1-x}.$$

Expanding the second logarithmic term results in an integral which is in part expressible as $s_{2n,2}$ i.e.

$$\mathcal{J}_1(2n+1) + \mathcal{J}_2(2n+1) = \frac{1}{2} s_{2n,2} + \frac{1}{2(2n)!} \int_0^1 \frac{\ln^{2n}(x) \ln(1+x)}{1-x} dx.$$

Expansion of $\ln(1+x)/(1-x)$ in a power series gives

$$\frac{\ln(1+x)}{(1-x)} = \sum_{k=1}^{\infty} [\ln(2) + (-1)^{k+1} \{\psi(k/2+1) - \psi(k+1) + \ln(2)\}] x^k.$$

The remaining integral is then given by

$$\frac{1}{2(2n)!} \int_0^1 \frac{\ln^{2n}(x) \ln(1+x)}{1-x} dx = \frac{1}{2} \sum_{k=1}^{\infty} \frac{[\ln(2) + (-1)^{k+1} \{\psi(k/2+1) - \psi(k+1) + \ln(2)\}]}{(k+1)^{2n+1}}.$$

As in the case of $\mathcal{J}_2(2n+1)$ these infinite summations are expressible in terms of sums defined above and we get

$$\mathcal{J}_1(2n+1) = \frac{1}{2}[s_{2n,2} - \tilde{\sigma}_{2n,2}] - \mathcal{M}(2n+1).$$

More generally, using the same methods one can show for arbitrary r that

$$\begin{aligned} \mathcal{J}_1(r) &= \frac{1}{2}[s_{r-1,2} - \tilde{\sigma}_{r-1,2}] - \mathcal{M}(r), \\ \mathcal{J}_2(r) &= \frac{1}{2}[s_{r-1,2} + \tilde{\sigma}_{r-1,2}] - \mathcal{C}(r) + 2^{-r-1}\zeta(r+1). \end{aligned}$$

The sum $\mathcal{C}(r)$ can also be written as the integral

$$\mathcal{C}(r) = \frac{(-1)^r}{2^{r+1}(r-1)!} \int_0^1 \frac{\ln^{r-1}(x) \log(1-x)}{x(1-x)} dx,$$

using the methods used above in connection with the Jordan sums. This integral can then be expressed in terms of the Nielsen integrals as

$$\mathcal{C}(r) = \frac{1}{2^{r+1}} (s_{r,1} + s_{r-1,2}) = \frac{1}{2^{r+1}} [\zeta(r+1) + s_{r-1,2}].$$

As a result of the latter simplified expression, $\mathcal{J}_2(r)$ can be written as

$$\mathcal{J}_2(r) = \frac{1}{2} [(1 - 2^{-r}) s_{r-1,2} + \tilde{\sigma}_{r-1,2}].$$

In the case of the Milgram sum $\mathcal{M}(r)$, it does not appear to be possible to express it in terms of the Nielsen integrals.

A.2 Approximate values for the alternating Euler sums

Since it does not appear to be possible to express the Jordan sums of odd order as closed form expressions and thus the corresponding alternating Euler sums, we have sought approximations for the later quantities. Beginning with the relation

$$\int_0^1 \frac{[Li_p(-t) - Li_p(-1)]}{1-t} dt = - \int_0^1 \frac{Li_p(-t) \ln(1-t)}{t} dt, \quad (17)$$

expansion of $Li_p(-t)$ on the rhs of this equation followed by integration with respect to t gives the alternating series $S_-(p)$ i.e.

$$\int_0^1 \frac{[Li_p(-t) - Li_p(-1)]}{1-t} dt = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} [\psi(k+1) + \gamma] = S_-(p). \quad (18)$$

If within the lhs of Eq. (17) $Li_p(-t)$ is expanded in powers of $t-1$, we get the relation

$$\int_0^1 \frac{Li_p(-t) - Li_p(-1)}{1-t} dt = - \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^1 (t-1)^{k-1} dt \sum_{j=1}^k S_k^{(j)} (2^{1+j-p} - 1) \zeta(p-j), \quad (19)$$

with the help of formulas for the derivatives [20] of the polylogarithm functions i.e.

$$\left[\frac{d^k Li_p(-t)}{dt^k} \right]_{t=1} = \sum_{j=1}^k S_k^{(j)} (2^{1+j-p} - 1) \zeta(p-j),$$

and where $S_k^{(j)}$ are the Stirling numbers of the first kind [22]. Integrating over t in Eq. (19) we get

$$S_-(p) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot k!} \sum_{j=1}^k S_k^{(j)} (2^{1+j-p} - 1) \zeta(p-j).$$

If the sum on the right-hand side of this equation is truncated to order $k = k_t$, we obtain a value of the *alternating* Euler sum which can be made accurate to

any desired number of decimals places. For example taking $p = 5$ and $k_t = 10$, an approximate value of the *alternating* sum is

$$S_-(5) \approx \sum_{k=1}^{10} \frac{(-1)^{k+1}}{k \cdot k!} \sum_{j=1}^k S_k^{(j)} (2^{j-4} - 1) \zeta(5 - j),$$

which when written out explicitly takes on the form

$$S_-(5) \approx -\frac{24387227}{1741824000} - \frac{358039}{11197440} \pi^2 - \frac{1968329}{130636800} \pi^4 + \frac{2152309}{3456000} \zeta(3) + \frac{1874237}{14515200} \ln(2),$$

an expression which is accurate to nine decimal places.

It is hoped that future work in this area will produce exact or more accurate values of the Euler alternating sums.

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